

Enhanced transmission due to disorder

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The transmissivity of a one-dimensional random system that is periodic on average is studied. It is shown that the transmission coefficient for frequencies corresponding to a gap in the band structure of the average periodic system increases with increasing disorder when the disorder is weak enough. This property is shown to be universal, independent of the type of fluctuations causing the randomness. In the case of strong disorder the transmission coefficient for frequencies in allowed bands is found to be a nonmonotonic function of the strength of the disorder. An explanation for the latter behavior is provided.

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It is well known today [1] that in the propagation of a classical wave through a one-dimensional periodic structure of finite length the amplitude of the transmitted wave decreases exponentially with increasing length of the system when the frequency of the wave is in a gap in the photonic band structure of the infinite lattice of the same period. If this periodic structure is now randomly disordered in such a way that it remains periodic on average, it might seem as if a wave whose frequency lies in the gap would be even more strongly attenuated, due to the additional random scattering. In this paper we show, on the basis of three different models, that in fact the opposite occurs: the transmissivity of the disordered system is larger than that of the periodic structure, when the frequency of a wave propagating through it falls in a band gap of the latter. A possible reason for this counterintuitive result will be presented.

We consider a scalar wave $U(z)$ that satisfies the one-dimensional Helmholtz equation

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c^2} \epsilon(z) \right] U(z) = 0, \quad (1)$$

where $\epsilon(z)$ is a random, periodic on average, function inside the disordered region $z \in (0, L)$ and is equal to ϵ_0 for $z \notin (0, L)$. In the case when a plane wave of a frequency ω is incident on the segment $(0, L)$ from the left, the solution of Eq. (1) for $z \notin (0, L)$ can be written in the form

$$U(z) = \begin{cases} e^{ik_0 z} + r(L)e^{-ik_0 z} & \text{for } z < 0 \\ t(L)e^{ik_0 z} & \text{for } z > L, \end{cases} \quad (2)$$

where $k_0 = \epsilon_0^{1/2} \omega / c$, which defines the reflection and the transmission coefficients $r(L)$ and $t(L)$, respectively.

The first model of a disordered structure that we consider is an alternating array of $2N$ dielectric slabs of di-

electric constants ϵ_1 and ϵ_2 in the region $(0, L)$. The width of the i th slab is given by $a_i = a(1 + \Delta_i)$, where the Δ_i are independent random variables that are uniformly distributed in the interval $(-\Delta, \Delta)$, where clearly $0 \leq \Delta < 1$. This structure is imbedded in a homogeneous medium and a wave of frequency ω is incident normally on it from the left. The complex transmission coefficient t has been calculated as a function of the length L of the disordered structure by means of a transfer matrix approach [2], for two frequencies of the incident wave in the vicinity of the lower edge of the lowest frequency band gap in the band structure of the average periodic lattice. For the parameters assumed ($\epsilon_1 = 2, \epsilon_2 = 3.5$) the frequencies of the lower and higher frequency edges

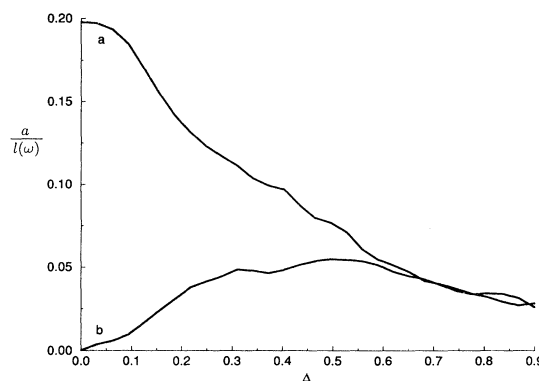


FIG. 1. Liapunov exponent as a function of the disorder parameter Δ for the first model with “positional” disorder. Curve a corresponds to the wave frequency $\omega a/c = 0.90$ above the lower frequency edge of the first band gap ($\omega_- a/c = 0.873$) and b corresponds to the frequency $\omega a/c = 0.85$ below the edge.

of this band gap are defined by $\omega_-a/c = 0.873$ and $\omega_+a/c = 1.038$, respectively. In Fig. 1 we plot the large- L limit of $-(a/L)\langle \ln T \rangle = a/l(\omega)$ as a function of the disorder parameter Δ for two different frequencies ω of the incident wave. Here $T = |t|^2$ is the transmissivity of the structure, the angular brackets denote an average over 100 realizations of the structure, and $l^{-1}(\omega)$ is the Liapunov exponent.

From the results presented in Fig. 1 we see that $l^{-1}(\omega)$ decreases monotonically with increasing Δ for the frequency of the electromagnetic wave within the band gap, which in turn means that the transmittance of the structure increases with increasing disorder. It is as if in the presence of the disorder, channels for the propagation of waves in this frequency range that are closed in the absence of the disorder open up due to the partial filling of the density of photonic states in the gap region by the tails of this density of states from the higher and the lower frequency bands bordering the gap. However, for the frequency in the allowed band, we obtain a surprising nonmonotonic dependence of the Liapunov exponent on the strength of the disorder Δ . The initial increase of $l^{-1}(\omega)$ with increasing Δ is readily understood as the suppression of the transmission in this frequency range due to the multiple scattering of the incident wave caused by the disorder. The decrease in the value of $l^{-1}(\omega)$ for strong disorder can be explained by a rather simple argument that will be presented below. The merging of the curves $l^{-1}(\omega)$ for all values of ω for values of Δ larger than about 0.6 is a reflection of the fact that for such large values of Δ the scattering structure has lost all traces of its underlying periodicity, as a consequence of which the band structure has disappeared, and waves of all frequencies experience the same disordered medium that is now homogeneous on average.

The second model of a disordered structure that we consider is a system of $2N - 1$ dielectric slabs, each of

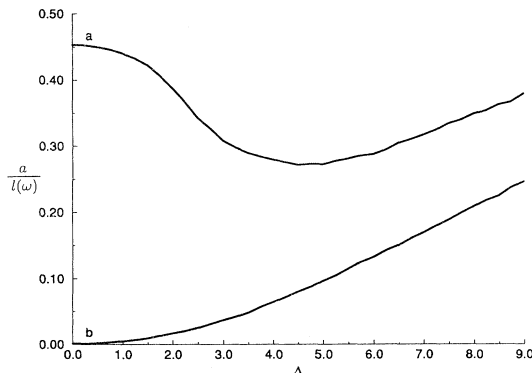


FIG. 2. Liapunov exponent as a function of the disorder parameter Δ for the second model with random ϵ_i . Curve *a* corresponds to the wave frequency $\omega a/c = 0.6$ above the lower frequency edge of the first band gap and *b* corresponds to the frequency $\omega a/c = 0.5$ below the edge. For the parameters of the second model assumed ($\epsilon = 9.0$) the position of the lower frequency edge of the first band gap is at $\omega_-a/c = 0.568$.

thickness a , in which the dielectric constant ϵ_i of i th slab is given by $\epsilon_{2i-1} = \epsilon + \delta\epsilon_i$, $\epsilon_{2i} = 1$, where the $\delta\epsilon_i$ are independent random variables that are uniformly distributed in the interval $[-\Delta, \Delta]$. In contrast to the first model of a disordered structure considered, in which the disorder parameter Δ was limited by physical considerations to the values $0 \leq \Delta < 1$, there is no restriction on the (positive) values that Δ can take in the present model. In Fig. 2 we plot the dependence of the Liapunov exponent $l^{-1}(\omega)$ as a function of the disorder parameter Δ when the frequency of the incident wave is in the lowest frequency gap of the photonic band gap structure of the average periodic system and when it is in the allowed band below it. If the frequency is in the band gap, we see that when Δ is increased from zero, $l^{-1}(\omega)$ initially decreases, indicating that the transmittance $T = |t|^2$ is increasing with increasing disorder. However, when Δ reaches a value of approximately 4.5, $l^{-1}(\omega)$ begins to increase monotonically with a further increase of Δ . In contrast, when the frequency of the incident wave is in the allowed band $l^{-1}(\omega)$ increases monotonically with increasing Δ .

The third model of a periodic system with disorder considered in the present paper is a continuous one: its dielectric constant is given by

$$\epsilon(z) = A \cos qz + \delta\epsilon(z), \quad (3)$$

where $\delta\epsilon(z)$ is a zero-mean, Gaussian random process, with the correlation function

$$\langle \delta\epsilon(z)\delta\epsilon(z') \rangle = \delta^2 \exp[-|z - z'|^2/l_{\text{cor}}^2]. \quad (4)$$

For a numerical calculation of the transmission coefficient we used the exact invariant imbedding equations [3,4]

$$\frac{dr(L)}{dL} = \frac{i}{2} k_0 \epsilon(L) [e^{-ik_0 L} + r(L)e^{ik_0 L}]^2, \quad (5a)$$

$$\frac{dt(L)}{dL} = \frac{i}{2} k_0 \epsilon(L) t(L) [1 + r(L)e^{2ik_0 L}], \quad (5b)$$

subject to the initial conditions $r(0) = 0$, $t(0) = 1$. The results are presented in Fig. 3.

When Figs. 1, 2, and 3 are compared, it is apparent that if the disorder is weak enough, the transmissivity for frequencies corresponding to gaps exhibits a rather unusual universal feature independent of the type of fluctuations: the transmission coefficient $T(\omega) \sim e^{-L/l(\omega)}$ increases when disorder appears ($\Delta \neq 0$) and continues to increase when the fluctuations grow. For strong disorder ($\Delta \sim 1$ in the first model, $\Delta > 4.5$ in the second one) the behavior of T as a function of Δ depends drastically on the type of disorder. Along with the expected decreasing of T in the second and third models one can see a quite surprising increase of the transmissivity of the system with “positional” discrete disorder (first model) when the strength of disorder Δ increases.

To understand the differences of the dependence of $l^{-1}(\omega)$ on the parameter Δ in the two discrete models studied in the present work, let us consider the disordered segment $(0, L)$ as a set of random scatterers, each

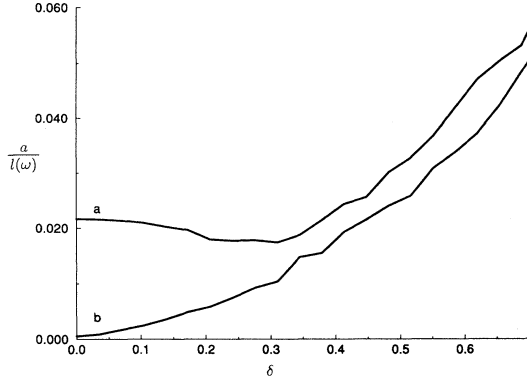


FIG. 3. Liapunov exponent as a function of the root mean square of the random function $\delta\epsilon(z)$ in the third, continuous, model. Curves *a* and *b* correspond to wave frequencies above and below the lower frequency edge of the first band gap, respectively.

of which is a dielectric slab with a random width (first model) or a random dielectric constant ϵ_i (second model) that possesses random reflection and transmission amplitudes R_{ind} and T_{ind} , respectively. In addition to the randomness in the individual scattering characteristics, we

have in the first model random distances $\delta_i = a(1 + \Delta_{2i})$ between consecutive scatterers. Then the transfer matrix \vec{M} for the entire system can be written as the product $\vec{M} = \vec{M}_1 \vec{F}_1 \vec{M}_2 \vec{F}_2 \cdots \vec{M}_{N-1} \vec{F}_{N-1} \vec{M}_N$, where

$$\vec{M}_i = \begin{pmatrix} \alpha_i & \beta_i \\ \beta_i^* & \alpha_i^* \end{pmatrix}, \quad \vec{F}_i = \begin{pmatrix} e^{ik\delta_i} & 0 \\ 0 & e^{-ik\delta_i} \end{pmatrix} \quad (6)$$

are the transfer matrices for the *i*th scattering slab and for the homogeneous space between consecutive scattering slabs, respectively. The effect of the interfaces in this system is incorporated into the definition of the matrices $\{\vec{M}_i\}$. The 11-element of the product $\vec{M}\vec{M}^+$ is related to the transmissivity of the structure $T = |t|^2$ by

$$\left(\vec{M}\vec{M}^+\right)_{11} = \frac{1}{T} - 1. \quad (7)$$

In averaging the matrix $\vec{M}\vec{M}^+$ we can use the independence of the matrices with different subscripts. This means that in the first model we can average over the distance between the $(N-1)$ th and N th scatterers independently of all the other scatterers:

$$\begin{aligned} \left\langle \vec{F}_{N-1} \vec{M}_{N-1} \vec{M}_N^+ \vec{F}_{N-1}^+ \right\rangle &= \begin{pmatrix} \langle |\alpha_N|^2 + |\beta_N|^2 \rangle & \langle 2\alpha_N \beta_N e^{i2k\delta_{N-1}} \rangle \\ \langle 2\alpha_N^* \beta_N^* e^{-i2k\delta_{N-1}} \rangle & \langle |\alpha_N|^2 + |\beta_N|^2 \rangle \end{pmatrix} \\ &= \begin{pmatrix} \left\langle \frac{1+R_N}{T_N} \right\rangle & 2\langle \alpha_N \beta_N \rangle \langle e^{i2k\delta_{N-1}} \rangle \\ 2\langle \alpha_N \beta_N \rangle^* \langle e^{-i2k\delta_{N-1}} \rangle & \left\langle \frac{1+R_N}{T_N} \right\rangle \end{pmatrix}. \end{aligned} \quad (8)$$

If the disorder is strong enough ($2\Delta ka \gg 1$), we can neglect the off-diagonal terms on the right-hand side of Eq. (8). By repeating this procedure for each of the random transfer matrices, we obtain

$$\left\langle \vec{M}\vec{M}^+ \right\rangle = \left\langle \frac{1+R_{\text{ind}}}{T_{\text{ind}}} \right\rangle^N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (9)$$

From a comparison of Eqs. (7) and (9) we find that

$$\left\langle \frac{1}{T} \right\rangle = \frac{1}{2} \left[1 + \left\langle \frac{1+R_{\text{ind}}}{T_{\text{ind}}} \right\rangle^N \right]. \quad (10)$$

For a long system ($N \rightarrow \infty$) Eq. (10) becomes approximately

$$\left\langle \frac{1}{T} \right\rangle \cong \frac{1}{2} \exp \left\{ N \ln \left\langle \frac{1+R_{\text{ind}}}{1-R_{\text{ind}}} \right\rangle \right\}, \quad (11)$$

which means that the localization length $l(\omega)$ is of the order of

$$\frac{a}{l(\omega)} = -\frac{1}{N} \langle \ln T \rangle \cong \ln \left\langle \frac{1+R_{\text{ind}}}{1-R_{\text{ind}}} \right\rangle. \quad (12)$$

In obtaining this result we have approximated $\langle \ln T \rangle$ by $-\ln(1/T)$ and have ignored questions of the self-averaging or non-self-averaging of the latter quantity to obtain a simple qualitative result. For weak individual scattering we obtain

$$\frac{a}{l(\omega)} \approx 2 \langle R_{\text{ind}} \rangle. \quad (13)$$

Therefore in the case of strong disorder the transmission coefficient of a disordered random system is determined completely by the mean value of the reflection coefficient

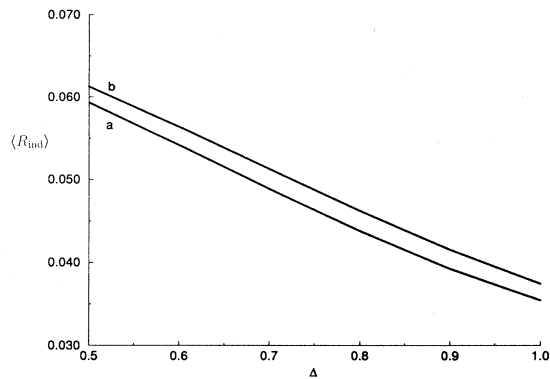


FIG. 4. Plot of the average value of the random reflection amplitude $\langle R_{\text{ind}} \rangle$. Curves *a* and *b* were calculated for the frequencies that correspond to the frequencies of the curves *a* and *b* in Fig. 1.

of a single scatterer. In the first model R_{ind} is a periodic function of the width d of the single slab and the averaging in Eq. (13) means just the integration of R_{ind} over d in the interval $[a(1 - \Delta), a(1 + \Delta)]$. It is clear that for

$\Delta \sim 1$ (strong “positional” disorder) the increase of the interval of integration of a periodic function causes the decrease of $\langle R_{\text{ind}} \rangle$. In the second model (random ϵ_i) the increase of Δ enhances the strength of a single scatterer and obviously leads to an increase of $\langle R_{\text{ind}} \rangle$.

We have carried out independent numerical calculations of $\langle R_{\text{ind}} \rangle$ (Fig. 4). The results are consistent with the reasoning above and provide an explanation for the behavior of $l^{-1}(\omega)$ for strong disorder (large Δ) depicted in Fig. 1.

In conclusion, we have shown on the basis of three different models of a one-dimensional random structure that is periodic on average that the transmissivity of the structure for waves with frequencies corresponding to a gap in the band structure it possesses in the absence of the randomness is increased by the randomness. In addition, we have given a qualitative explanation of the observed nonmonotonic dependence of the length on the disorder parameter Δ for waves whose frequencies are in an allowed band.

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